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Invariants and Radiation of Some Nonstationary Systems

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Abstract

In this paper formulas are obtained by means of the coherent-state method for calculating the radiation power of a nonstationary quantum system of N charged particles whose Hamiltonian is a general quadratic form with respect to coordinates and momenta. The transitions between the coherent states and the Fock states of this system are discussed. The radiation is calculated both in the dipole approximation and strictly. As an example, the radiation of a charge in homogeneous varying electric and magnetic fields is found. The classical limit is considered.

1. *Introduction*

The radiation of nonstationary classical and quantum systems and the connection between quantum and classical formulas for calculating the radiation have been discussed by Schwinger (1949, 1954, 1973). In considering the radiation of nonstationary quantum systems, the coherent-state representation (Glauber, 1963a, b, c) proves to be the most effective that enables us to visualize the interconnection between quantum and classical approach to the radiation of the system. As the coherent states describe the wave packet moving along the classical trajectory in the phase space, the quantum coherent-state method is close to the classical investigation. The coherent-state method is extremely convenient in calculating the radiation of nonstationary systems whose Hamiltonian is a general quadratic form with respect to coordinates and momenta. The coherent and Fock states and the Green's function for these systems have been determined by Malkin et al., (1971) and Dodonov et al. (1974).

Below the formulas are obtained for calculating the radiation power of the system of N charged particles with the nonrelativistic nonstationary quadratic Hamiltonian. It should be noted, however, that the use of the proper-time representation gives us the possibility of employing the method developed below not only for nonrelativistic but for relativistic systems.

Earlier the problem of the radiation of nonstationary quadratic systems has been considered partially in Ivanova et al. (1974; 1975b). The radition of the system with the stationary quadratic Hamiltonian of the special form has been discussed in Ivanova et al. (1975a).

In this paper the radiation is calculated according to perturbation theory to the first order in the magnitudes of the charges. It is necessary to point out, however, that the coherent-state method allows us to evaluate the dipole radiation of an arbitrary nonstationary quadratic system not only taking into account the perturbation theory but exactly, since the system of charges interacting with the electromagnetic field in the dipole approximation is described by a quadratic Hamiltonian. The formulas given below may be obtained from the exact transition amplitudes to the first order in the magnitudes of the charges.

2. *Radiation of an Arbitrary Time-Dependent Quadratic System*

We consider a quantum system of N charged particles whose Hamiltonian is quadratic with respect to coordinates $\mathbf{q}^a$ and momenta $\mathbf{p}^a$ of particles (Holz, 1970; Malkin et al., 1971, 1973):

$$H_0(t) = \sum_{a,b,i,j} [(B_1)_{ij}^{ab} p_i^a p_j^b + (B_2)_{ij}^{ab} p_i^a q_j^b + (B_3)_{ij}^{ab} q_i^a p_j^b + (B_4)_{ij}^{ab} q_i^a q_j^b + (C_1)_i^a p_i^a + (C_2)_i^a q_i^a] \qquad (\hbar = c = 1) \tag{2.1a}$$

Here B_{ij}^{ab} and C_i^a are arbitrary functions of time. C_i^a are real numbers and $B_{ij}^{ab} = B^{*ba}_{ji}$ (* means a complex conjugation). The superscripts designate the particles and run over $1, 2, \ldots, N$, and the subscripts are the usual Cartesian coordinates of three-dimensional vectors.

Let us introduce the $3N$ vector $\mathbf{q} = \{q_j\}$ with the subscript running over $1, 2 \ldots, 3N$ and constructed corresponding to the rule ${}^T\mathbf{q} = (\mathbf{q}^1, \ldots, \mathbf{q}^a, \ldots, \mathbf{q}^N)$ and $3N$ vector ${}^T\mathbf{p} = (\mathbf{p}^1, \ldots, \mathbf{p}^a, \ldots, \mathbf{p}^N)$, the index T representing the transposition operation. The rule of the abbreviated record is: If the the superscript is present, the subscript runs over 1, 2, 3; if the superscript is absent, the subscript runs over $1, 2, \ldots, 3N$. Because the summation over the superscripts and subscripts is always independent, it is possible to rewrite the Hamiltonian $H_0(t)$ in the matrix form:

$$H_0(t) = \mathbf{Q}B\mathbf{Q} + \mathbf{C}\mathbf{Q} \tag{2.1b}$$

where $Q_j = p_j$, $Q_{3N+j} = q_j$, B is a $6N \times 6N$ matrix, and $\mathbf{C}$ is a $6N$ vector:

$$\mathbf{Q} = \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix}, \qquad B \equiv \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \qquad \mathbf{C} \equiv \begin{pmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{pmatrix}$$

Let the Hamiltonian (2.1) be stationary before the moment of time t_{in}. We designate this stationary Hamiltonian as H_{in}. Let the Hamiltonian $H_0(t)$

be stationary after the moment of time t_f again. We designate it as H_f. In particular, $t_{\text{in}} \to -\infty$, $t_f \to \infty$ may be. In what follows the subscript f stands for the values corresponding to the final Hamiltonian H_f, and the subscript in stands for the values corresponding to the initial Hamiltonian H_{in}.

In (Malkin et al., 1971; 1973) the $6N$ Hermitian linear time-dependent invariants have been constructed for the Hamiltonian $H_0(t)$:

$$\mathbf{I}(t) = \Lambda(t)\mathbf{Q} + \boldsymbol{\delta}(t), \qquad i\frac{\partial}{\partial t}\mathbf{I}(t) = [H_0, \mathbf{I}(t)] \tag{2.2}$$

where

$$\Lambda \equiv \begin{pmatrix} \Lambda_1 & \Lambda_2 \\ \Delta_3 & \Lambda_4 \end{pmatrix} = \tilde{T}\exp\left[i \int_{t_{\text{in}}}^{t_f} d\tau \sigma_2 (B + B^*)\right]$$

$$\boldsymbol{\delta} \equiv i \int_{t_{\text{in}}}^{t_f} d\tau \Lambda \sigma_2 \mathbf{C}, \qquad \sigma_2 = i\begin{pmatrix} 0_{3N} & -E_{3N} \\ E_{3N} & 0_{3N} \end{pmatrix} \tag{2.3}$$

here E_{3N} is the $3N \times 3N$ unit matrix; $\tilde{T}$ stands for the antichronological product. The linear transformation (2.2) is canonical. Matrix Λ is the real symplectic matrix and, hence, $\Lambda\sigma_2\, {}^T\Lambda = \sigma_2$.

We can introduce, instead of $\mathbf{I}(t)$, lowering and raising operators $A_j(t)$, $A_j^\dagger(t)$:

$$A_j(t) = (1/\sqrt{2})[iI_j(t) + I_{3N+j}(t)] \tag{2.4}$$

$$[A_j, A_k^\dagger] = \delta_{jk}, \qquad j, k = 1, 2, \ldots, 3N$$

Using (2.2) we rewrite (2.4) in the form

$$A_j(t) = \frac{1}{\sqrt{2}} \sum_k [(\lambda_p)_{jk} p_k + (\lambda_q)_{jk} q_k + \Delta_j] \tag{2.5}$$

where

$$\lambda_p = \Lambda_3 + i\Delta_1, \qquad \lambda_q = \Lambda_4 + i\Lambda_2, \qquad \Delta_j = i\delta_j + \delta_{3N+j} \tag{2.6}$$

In (Malkin et al., 1971; 1973) the coherent states $|\boldsymbol{\alpha}; t\rangle$ and the Fock states $|\mathbf{n}; t\rangle$ of a system with the Hamiltonian $H_0(t)$ have been found. The states $|\boldsymbol{\alpha}; t\rangle$ are eigenstates of lowering operators A_j and satisfy the Schrödinger equation:

$$A_j(t)|\boldsymbol{\alpha}; t\rangle = \alpha_j |\boldsymbol{\alpha}; t\rangle, \qquad \left(i\frac{\partial}{\partial t} - H_0\right)|\boldsymbol{\alpha}; t\rangle = 0 \tag{2.7}$$

$|\boldsymbol{\alpha}; t\rangle$ are expressed in terms of $\Lambda(t)$ and $\boldsymbol{\delta}(t)$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{3N})$ being constant complex numbers. The real and imaginary parts of the vector $\boldsymbol{\alpha}$ represent correspondingly the initial average of coordinates and momenta. The coherent states $|\boldsymbol{\alpha}; t\rangle$ are the generating functions (Glauber, 1963a, b, c) for the Fock states $|\mathbf{n}; t\rangle$. The states $|\mathbf{n}; t\rangle$ are the eigenstates of the invariants $A_j^\dagger A_j$. They are orthonormalized and satisfy the Schrödinger equation:

$$|\mathbf{n}; t\rangle \equiv \prod_j |n_j; t\rangle, \qquad n_j = 0, 1, 2, \ldots$$

$$\left(i\frac{\partial}{\partial t} - H_0\right)|\mathbf{n}; t\rangle = 0, \qquad \langle \mathbf{m}; t|\mathbf{n}; t\rangle = \delta_{\mathbf{m},\mathbf{n}}$$

$$A_j^\dagger A_j|\mathbf{n}; t\rangle = n_j|\mathbf{n}; t\rangle \tag{2.8}$$

$$A_j|n_j; t\rangle = (n_j)^{1/2}|n_j - 1; t\rangle, \qquad A_j^\dagger|n_j; t\rangle = (n_j + 1)^{1/2}|n_j + 1; t\rangle$$

$|\mathbf{n}; t\rangle$ are expressed in terms of Hermite polynomials of $3N$ variables (Erdélyi, 1953).

Let the integrals of motion $\mathbf{I}_{\text{in}} = \Lambda_{\text{in}}\mathbf{Q} + \boldsymbol{\delta}_{\text{in}}$ and the coherent states $|\boldsymbol{\alpha}; \text{in}\rangle$ correspond to the stationary initial Hamiltonian H_{in}, and the integrals of motion $\mathbf{I}_f = \Lambda_f\mathbf{Q} + \boldsymbol{\delta}_f$ and the coherent states $|\boldsymbol{\beta}; f\rangle$ correspond to the stationary final Hamiltonian H_f. As the initial (final) Hamiltonian is quadratic with respect to $\mathbf{I}_{\text{in}}(\mathbf{I}_f)$, it is possible to transform the integrals of motion $\mathbf{I}_{\text{in}}$ to the new integrals of motion $\underset{\sim}{\mathbf{I}}_{\text{in}}$, using the linear canonical transformation in the form (2.2):

$$\mathbf{I}_{\text{in}} = \mathcal{G}\underset{\sim}{\mathbf{I}}_{\text{in}} + \mathcal{d} \tag{2.9}$$

The invariants $\underset{\sim}{\mathbf{I}}_{\text{in}}$ define the new raising and lowering operators $\underset{\sim}{A}_j^\dagger$, $\underset{\sim}{A}_j$ and the new coherent states $|\underset{\sim}{\boldsymbol{\alpha}}; \text{in}\rangle$. Sometimes one can choose a transformation (2.9) that reduces the Hamiltonian H_{in} to the canonical form

$$H_{\text{in}} = \sum_j (\Omega_{\text{in}})_j[(\underset{\sim}{A}_{\text{in}}^\dagger)_j(\underset{\sim}{A}_{\text{in}})_j + \tfrac{1}{2}] \tag{2.10}$$

the new coherent states $|\underset{\sim}{\boldsymbol{\alpha}}; \text{in}\rangle$ being the generating functions for the stationary states of H_{in}. The frequencies $(\Omega_{\text{in}})_j$ may be both positive and negative. Let us note that the fomula (2.10) makes it possible to find the spectrum of the system. For the particular potential it has been determined in Ivanova et al. (1975a).

The state $|\boldsymbol{\alpha}; t\rangle$ $(|\underset{\sim}{\boldsymbol{\alpha}}; t\rangle)$ is the time evolution of the state which at the initial moment of time coincides with the state $|\boldsymbol{\alpha}; \text{in}\rangle(|\underset{\sim}{\boldsymbol{\alpha}}; \text{in}\rangle)$. The time evolution is determined by the evolution operator U_0 of the Hamiltonian $H_0(t)$ as $U_0(t, t_{\text{in}})|\boldsymbol{\alpha}; \text{in}\rangle = |\boldsymbol{\alpha}; t\rangle[U_0(t, t_{\text{in}})|\underset{\sim}{\boldsymbol{\alpha}}; \text{in}\rangle = |\underset{\sim}{\boldsymbol{\alpha}}; t\rangle]$. The explicit expression for U_0 is given in Malkin et al. (1971; 1973).

Let us consider the radiation of the system with the Hamiltonian (2.1). It should be noted that the radiation of the charges with the Hamiltonian H_0 at a moment of time $t > t_f$ has been discussed earlier in Ivanova et al. (1974), whereas the formulas obtained below are valid for an arbitrary moment. In case of $t > t_f$ these formulas will be considerably simplified and will transform to formulas of the paper by Ivanova et al. (1974).

The vector potential of the radiation field in the Schrödinger representation can be written in the form

$$\mathscr{A}_{\text{rad}}(\mathbf{q}^a) = \sum_{\lambda,\sigma} \left(\frac{2\pi}{V\omega_\lambda}\right)^{1/2} \mathbf{e}_{\lambda,\sigma} [b_{\lambda,\sigma} \exp(i\mathbf{k}_\lambda \mathbf{q}^a) + b^\dagger_{\lambda,\sigma} \exp(-i\mathbf{k}_\lambda \mathbf{q}^a)] \tag{2.11}$$

where $b^\dagger_{\lambda,\sigma}$ ($b_{\lambda,\sigma}$) are the boson raising (lowering) operators of a photon with the frequency ω_λ, the wave vector $\mathbf{k}_\lambda$, and with the polarization vector $\mathbf{e}_{\lambda,\sigma}$. We can rewrite $\mathscr{A}_{\text{rad}}(\mathbf{q}^a)$ in terms of invariants

$$b_{\lambda,\sigma}(t) = b_{\lambda,\sigma} \exp(i\omega_\lambda t) \quad \text{and} \quad b^\dagger_{\lambda,\sigma}(t) = b^\dagger_{\lambda,\sigma} \exp(-i\omega_\lambda t)$$

satisfying the following relations:

$$b_{\lambda,\sigma}(t)|n_{\lambda,\sigma}; t\rangle = (n_{\lambda,\sigma})^{1/2}|n_{\lambda,\sigma} - 1; t\rangle$$
$$b^\dagger_{\lambda,\sigma}(t)|n_{\lambda,\sigma}; t\rangle = (n_{\lambda,\sigma} + 1)^{1/2}|n_{\lambda,\sigma} + 1; t\rangle$$

where $\Pi_{\lambda,\sigma}|n_{\lambda,\sigma}; t\rangle \equiv |\mathbf{n}_{\text{rad}}; t\rangle$ are the stationary states of the radiation field. Let the initial state be $|\mathbf{n}_{\text{rad}}; t\rangle$ and the final state be $|\mathbf{m}_{\text{rad}}; t\rangle$. If we discuss the transition with the emission of one photon corresponding to the raising operator $b^\dagger_{\lambda,\sigma}$, we must suppose that $n_{\lambda,\sigma} + 1 = m_{\lambda,\sigma}$ and $n_{\lambda',\sigma'} + 1 \neq m_{\lambda'\sigma'}(\lambda', \sigma', \neq \lambda, \sigma)$. In case of all $n_{\lambda,\sigma}$ being equal to zero at the initial moment of time we deal with the spontaneous radiation.

The radiation field and the charged particles (2.1) are considered as the unified system whose Hamiltonian $H(t)$ is obtained from the Hamiltonian $H_0(t)$ by the substitution of $\mathbf{p}^a - e^a \mathscr{A}_{\text{rad}}(\mathbf{q}^a)$ for $\mathbf{p}^a$ (here e^a are the charges of the particles) and by the addition of the Hamiltonian of the free radiation field H_{rad}:

$$H(t) = H_0(t) + H_{\text{rad}} + H_{\text{int}}(t) \tag{2.12}$$

The interaction Hamiltonian $H_{\text{int}}(t)$ to the first order in the magnitudes of the charges can be written as follows:

$$H_{\text{int}} = -(\mathbf{a}B\mathbf{Q} + \mathbf{Q}B\mathbf{a} + C\mathbf{a}) \tag{2.13}$$

where the $6N$ vector ${}^T\mathbf{a} = ({}^T\mathscr{A}_{\text{rad}}, {}^T\mathbf{0}_{3N})$ is constructed from the vector potential of the radiation field as ${}^T\mathscr{A}_{\text{rad}} = (e^1\mathscr{A}^1_{\text{rad}}, \ldots, e^a\mathscr{A}^a_{\text{rad}}, \ldots, e^N\mathscr{A}^N_{\text{rad}})$, $\mathscr{A}_{\text{rad}}(\mathbf{q}^a) \equiv \mathscr{A}^a_{\text{rad}}$.

Let us introduce the notation $|\boldsymbol{\alpha}; \mathbf{n}_{\text{rad}}; t\rangle \equiv |\boldsymbol{\alpha}; t\rangle|\mathbf{n}_{\text{rad}}; t\rangle$($|\mathbf{n}; \mathbf{n}_{\text{rad}}; t\rangle \equiv |\mathbf{n}; t\rangle|\mathbf{n}_{\text{rad}}; t\rangle$) for the states of the unified system consisting of the charges and the radiation field in case of the interaction between the field and the charges being absent. These states at the initial moment of time are designated by $|\boldsymbol{\alpha}; \mathbf{n}_{\text{rad}}; \text{in}\rangle$. The wave function of the Hamiltonian (2.12) at an arbitrary moment t can be written as $U(t, t_{\text{in}})|\boldsymbol{\alpha}; \mathbf{n}_{\text{rad}}; \text{in}\rangle$, where U is the evolution operator of $H(t)$.

The transition amplitude $\mathcal{M}_{\beta,\alpha}(t)$ between the coherent states $|\boldsymbol{\alpha}; \text{in}\rangle$ and $|\boldsymbol{\beta}; f[t]\rangle$ at a moment t is as follows:

$$\mathcal{M}_{\beta,\alpha}(t) = \langle \boldsymbol{\beta}; \mathbf{m}_{\text{rad}}; f[t] \,|\, U(t, t_{\text{in}}) |\boldsymbol{\alpha}; \mathbf{n}_{\text{rad}}; \text{in}\rangle \tag{2.14}$$

The transition probability between such states is $|\mathcal{M}_{\beta,\alpha}(t)|^2$. The amplitude $\mathcal{M}_{\beta,\alpha}$ to the first order in charges is

$$\mathcal{M}_{\beta,\alpha}(t) = -i \int_{t_0}^{t} \langle \boldsymbol{\beta}; \mathbf{m}_{\text{rad}}; f[t] \,|\, U_0(t, \tau)\, U_{\text{rad}}(t, \tau) \times H_{\text{int}}(\tau) U_0(\tau, t_{\text{in}})\, U_{\text{rad}}(\tau, t_{\text{in}}) |\boldsymbol{\alpha}; \mathbf{n}_{\text{rad}}; \text{in}\rangle \, d\tau, \qquad |\mathbf{n}_{\text{rad}}; t\rangle \neq |\mathbf{m}_{\text{rad}}; t\rangle \tag{2.15}$$

where U_{rad} is the time evolution operator of the Hamiltonian H_{rad} and t_0 is the moment of time at which the charges and the radiation field begin to interact $(t > t_0 \geqslant t_{\text{in}})$.

Introducing the $6N$ vectors $\mathbf{R}$ and $\mathbf{F}$ and the $6N \times 6N$ matrix L as follows:

$$\mathbf{R} = \begin{pmatrix} \mathbf{A} \\ \mathbf{A}^+ \end{pmatrix}, \quad \mathbf{F} = \frac{1}{\sqrt{2}} \begin{pmatrix} \boldsymbol{\Delta} \\ \boldsymbol{\Delta}^* \end{pmatrix}, \qquad L^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} \lambda_p & \lambda_q \\ \lambda_p^* & \lambda_q^* \end{pmatrix}$$

one can obtain

$$\mathbf{Q} = L\mathbf{R} - L\mathbf{F} \tag{2.16}$$

Using the formula (2.16), we can present the exponential factors $\exp(\pm i\mathbf{k}\mathbf{q}^b)$ in the expression for the vector potential of the radiation field (2.11) as the Weyl unitary displacement operators $D_s{}^c(\kappa^b{}^c_s) = \exp[\kappa^b{}^c_s (A^+)_s{}^c - (\kappa^*)^b{}^c_s A_s{}^c]$:

$$\exp(-i\mathbf{k}\mathbf{q}^b) = \exp[\psi_1{}^b(t)] \prod_{c,s} D_s{}^c[\kappa^b{}^c_s(t)] \tag{2.17}$$

where

$$\kappa^{bc}_{\ s} = -\frac{1}{\sqrt{2}} \sum_j k_j (\lambda_p)^{cb}_{sj}, \qquad \psi_1{}^b = -i\,\text{Im} \sum_{j,s,c} k_j (\lambda_p^*)^{cb}_{sj} \Delta_s{}^c \tag{2.18}$$

The Weyl operators act upon the coherent states in the following way (Glauber, 1963a, b, c):

$$\prod_{c,s} D_s{}^c(\kappa^b{}^c_s) |\{\alpha_k{}^d\}; t\rangle = |\{\alpha_k{}^d + \kappa^b{}^d_k\}; t\rangle \exp(\psi_2{}^b) \tag{2.19}$$

where

$$\psi_2{}^b = i\,\text{Im} \sum_{c,s} \kappa^b{}^c_s (\alpha^*)_s{}^c, \quad |\{\alpha_k{}^d\}; t\rangle \equiv |\boldsymbol{\alpha}; t\rangle \tag{2.20}$$

It is easy to express the vector $\mathbf{Q}$ in terms of the vector $\mathbf{R}$ in the amplitude (2.15). Then we must use the obvious identity

$$U_0(t, \tau)I_\alpha(\tau)U_0(\tau, t_{\text{in}}) = I_\alpha(t)U_0(t, t_{\text{in}}), \qquad \alpha = 1, 2, \ldots, 6N \quad (2.21)$$

for the integrand of (2.15) and express the vector $\mathbf{R}$ in terms of the vector ${}^T\boldsymbol{\Upsilon} = ({}^T\mathbf{A}, {}^T\mathbf{A}_f^\dagger)$, $\mathbf{R}$ being

$$\mathbf{R} = L^{-1}M\boldsymbol{\Upsilon} - L^{-1}M\boldsymbol{\Gamma} + \mathbf{F} \tag{2.22}$$

here

$$\boldsymbol{\Gamma} = \frac{1}{\sqrt{2}}\begin{pmatrix}\boldsymbol{\Delta}\\ \boldsymbol{\Delta}_f^*\end{pmatrix}, \qquad M \equiv \begin{pmatrix} M_1 & M_2\\ M_3 & M_4\end{pmatrix}$$

$$M_1 = \sqrt{2}[\lambda_p - \lambda_q(\lambda_q^*)_f^{-1}(\lambda_p^*)_f]^{-1}, \qquad M_2 = \sqrt{2}[(\lambda_p^*)_f - (\lambda_q^*)_f\lambda_q^{-1}\lambda_p]^{-1}$$
$$M_3 = \sqrt{2}[\lambda_q - \lambda_p(\lambda_p^*)_f^{-1}(\lambda_q^*)_f]^{-1}, \qquad M_4 = \sqrt{2}[(\lambda_q^*)_f - (\lambda_p^*)_f\lambda_p^{-1}\lambda_q]^{-1} \tag{2.23}$$

We shall consider now the spontaneous transitions between coherent states $|\boldsymbol{\alpha}; \text{in}\rangle$ and $|\boldsymbol{\beta}; f\rangle$ with the emission of a photon with the frequency ω, the wave vector $\mathbf{k}$, and the polarization vector $\mathbf{e}$. Let the averaging of the initial states of the charges be described by the density matrix

$$\rho_{\text{in}} = \int d^2\boldsymbol{\alpha} P_{\text{in}}(\boldsymbol{\alpha})|\boldsymbol{\alpha}; \text{in}\rangle\langle\boldsymbol{\alpha}; \text{in}|$$

where P_{in} is the Glauber P function (Glauber, 1963). Let the summation over the final states be described by the density matrix

$$\rho_f = \int d^2\boldsymbol{\beta} P_f(\boldsymbol{\beta})|\boldsymbol{\beta}; f\rangle\langle\boldsymbol{\beta}; f|$$

Then, using the form suggested by Schwinger (1949; 1954), we find the power radiated into a unit solid angle enclosing $\mathbf{n_k}$ (here $\mathbf{n_k}$ is the unit vector directed along $\mathbf{k}$), and contained in a unit frequency interval about the frequency ω as follows:

$$\mathscr{P}(\mathbf{n_k}, \mathbf{e}, \omega, t) = \frac{\omega^2}{4\pi^2}\int d\tau\, d^2\boldsymbol{\alpha}\, d^2\boldsymbol{\beta} P_{\text{in}}(\boldsymbol{\alpha}) \times P_f(\boldsymbol{\beta})e^{-i\omega\tau}J(t - \tfrac{1}{2}\tau, t)J^*(t + \tfrac{1}{2}\tau, t) \tag{2.24}$$

In the dipole approximation taking into account the commutator $[\mathbf{a}, \mathbf{Q}] = 0$, we obtain for $J_{\text{dip}}(\tau, t)$ the following expression:

$$J_{\text{dip}}(\tau, t) = [\boldsymbol{\zeta}W(\tau, t)\boldsymbol{\sigma} + \mathbf{h}(\tau, t)\boldsymbol{\sigma}]\langle\boldsymbol{\beta}; f[t]|\boldsymbol{\alpha}; t\rangle \tag{2.25}$$

where

$${}^TW(\tau, t) = -[B(\tau) + {}^TB(\tau)]L(\tau)L^{-1}(t)M(t)$$
$$\mathbf{h}(\tau, t) = -C(\tau) + [B(\tau) + {}^TB(\tau)]L(\tau)[L^{-1}(t)M(t)\boldsymbol{\Gamma}(t) + \mathbf{F}(\tau) - \mathbf{F}(t)] \tag{2.26}$$

$\boldsymbol{\zeta}$ and $\boldsymbol{\sigma}$ are the $6N$ vectors ${}^T\boldsymbol{\zeta} = ({}^T\boldsymbol{\alpha}, {}^T\boldsymbol{\beta}^*)$, ${}^T\boldsymbol{\sigma} = ({}^T\boldsymbol{\sigma}_{3N}, {}^T\mathbf{0}_{3N})$, $\boldsymbol{\sigma}_{3N}$ being the $3N$ vector ${}^T\boldsymbol{\sigma}_{3N} = (e^1\mathbf{e}, \ldots, e^a\mathbf{e}, \ldots, e^N\mathbf{e})$. It is easy to obtain the nonradiative transition amplitude $\langle\boldsymbol{\beta}; f[t]\,|\boldsymbol{\alpha}; t\rangle$ at an arbitrary moment of time t by evaluating the corresponding Gaussian integral:

$$\langle\boldsymbol{\beta}; f[t]\,|\boldsymbol{\alpha}; t\rangle = \langle\mathbf{0}; f[t]\,|\,\mathbf{0}; t\rangle \exp\left[-\tfrac{1}{2}(|\boldsymbol{\alpha}|^2 + |\boldsymbol{\beta}|^2) + \mathbf{s}\boldsymbol{\zeta} - \tfrac{1}{2}\boldsymbol{\zeta}S\boldsymbol{\zeta}\right] \quad (2.27)$$

where

$$s_1 = (1/\sqrt{2})(\boldsymbol{\Delta}^* - \lambda_p^*\lambda_p^{-1}\boldsymbol{\Delta}) + (i/2)\{[{}^TM_3(\lambda_p^*)_f^{-1} - {}^T\lambda_p^{-1}M_4]\boldsymbol{\Delta}_f^*$$
$$- ({}^TM_3\lambda_p^{-1} + {}^T\lambda_p^{-1}M_3)\boldsymbol{\Delta}\}$$
$$s_2 = (1/\sqrt{2})[\boldsymbol{\Delta}_f - (\lambda_p)_f(\lambda_p^*)_f^{-1}\boldsymbol{\Delta}_f] + (i/2)\{[(\lambda_p^\dagger)_f^{-1}M_4 + {}^TM_4(\lambda_p^*)_f^{-1}\boldsymbol{\Delta}_f^*$$
$$+ [(\lambda_p^\dagger)_f^{-1}M_3 - {}^TM_4\lambda_p^{-1}]\boldsymbol{\Delta}\}$$

$$\begin{aligned} S_1 &= -{}^T\lambda_p^{-1}(i\sqrt{2}M_3 + \lambda_p^\dagger), \qquad S_2 = -i\sqrt{2}\,{}^T\lambda_p^{-1}M_4 \\ S_3 &= i\sqrt{2}(\lambda_p^\dagger)_f^{-1}M_3, \qquad S_4 = (\lambda_p^\dagger)_f^{-1}[i\sqrt{2}M_4 - {}^T(\lambda_p)_f] \end{aligned} \quad (2.28)$$

The amplitude $\langle\mathbf{0}; f[t]\,|\,\mathbf{0}; t\rangle$ is

$$\langle\mathbf{0}; f[t]\,|\,\mathbf{0}; t\rangle = 2^{3N/4}\{\det[i(\lambda_p^*)_f M_3^{-1}]\}^{-1/2} \exp\{-(|\boldsymbol{\Delta}|^2 + |\boldsymbol{\Delta}_f|^2)/4$$
$$- i\int_{t_{\rm in}}^{t} \mathrm{Im}(\boldsymbol{\Delta}\dot{\boldsymbol{\Delta}}^* - \boldsymbol{\Delta}_f\dot{\boldsymbol{\Delta}}_f^*)d\tau/2 + (i/4)[\boldsymbol{\Delta}(\lambda_p^* + \sqrt{2}\,{}^TM_3)\lambda_p^{-1}\boldsymbol{\Delta}$$
$$+ \boldsymbol{\Delta}_f^*(\lambda_{pf} + \sqrt{2}\,{}^TM_4)(\lambda_p^*)_f^{-1}\boldsymbol{\Delta}_f^* + \sqrt{2}\boldsymbol{\Delta}({}^T\lambda_p^{-1}M_4$$
$$- {}^TM_3(\lambda_p^*)_f^{-1})\boldsymbol{\Delta}_f^*]\} \quad (2.29)$$

The amplitude (2.27) is the generating function of the amplitudes $\langle\mathbf{m}; f[t]\,|\,\mathbf{n}; t\rangle$ (Malkin et al., 1971; 1973) which are expressed in terms of Hermite polynomials of $6N$ variables (Erdélyi, 1953).

Let us consider the spontaneous transitions between the states of the radiation field $|\nu_\omega; t\rangle\Pi'_\lambda|n_\lambda; t\rangle$ and $|\mu_\omega; t\rangle\Pi'_\lambda|m_2; t\rangle$ $(\mathbf{e}_{\lambda,\sigma} \equiv \mathbf{e}_\lambda)$, where the prime means that the stationary state of the radiation field corresponding to the frequency ω is absent, and $|\nu_\omega; t\rangle$ and $|\mu_\omega; t\rangle$ are the coherent states of the radiation field corresponding to the frequency ω. Then the integrand of the expression for the radiation power (2.24) will contain the additional multiplier

$$J_{\rm ad}(\tau, t) = [|\mu_\omega|^2 e^{-i\omega\tau} + |\nu_\omega|^2 e^{i\omega\tau} + 2\mathrm{Re}(\mu_\omega\nu_\omega e^{-i2\omega t})]\exp(-|\mu_\omega - \nu_\omega|^2).$$

If the Hamiltonian H_0 is stationary, the formula (2.25) has the very simple form:

$$J_{\rm dip}^{\rm st}(\tau, t) \equiv J_{\rm dip}^{\rm st}(\tau) = i\sqrt{2}(\boldsymbol{\alpha}\dot{\lambda}_p^*\boldsymbol{\sigma}_{3N}$$
$$- \boldsymbol{\beta}^*\dot{\lambda}_p\boldsymbol{\sigma}_{3N})\exp[-(|\boldsymbol{\alpha}|^2 + |\boldsymbol{\beta}|^2)/2 + \boldsymbol{\alpha}\boldsymbol{\beta}^*] \quad (2.25')$$

(Here we consider $\mathbf{C} = 0$). In this case the expression (2.24) for calculating the radiation power in the dipole approximation transforms into its classical limit when

$$P_{\text{in}}(\boldsymbol{\alpha}) = \delta^{(2)}(\boldsymbol{\alpha} - \boldsymbol{\alpha}_0), \qquad P_f = \pi^{-3N}, |\alpha_{0j}| \gg 1$$

here $\delta^{(2)}$ is the delta function and the vector $\boldsymbol{\alpha}_0$ gives the intial point of the charges in the phase space of the system.

Let us determine $J(\tau, t)$ in the general case without being bound to the dipole approximation. Taking into account the equality

$$\exp(\mp i\mathbf{k}_\lambda \mathbf{q}^b)\mathbf{p}^a = (\mathbf{p}^a \pm \delta_{ab}\mathbf{k}_\lambda)\exp(\mp i\mathbf{k}_\lambda \mathbf{q}^b)$$

we rewrite the interaction Hamiltonian (2.13) in the form that allows us to use the formulas (2.17) and (2.19). The further calculation is analogous to the corresponding calculation in the dipole case. In that manner we arrive at the following formula:

$$J(\tau, t) = \sum_{a,b,i,j} \{[\alpha_i{}^a + \kappa^{ba}_i(\tau)][W_1(\tau, t)]^{ab}_{ij} + (\beta^*)_i{}^a [W_3(\tau, t)]^{ab}_{ij} + [h_1(\tau, t)]_j{}^b + [B_1(\tau)]^{bb}_{ji} k_i\} T^b_j(\tau, t) \tag{2.30}$$

where

$$T_j{}^b(\tau, t) = e^b(\mathbf{e})_j \exp[\psi_1{}^b(\tau) + \psi_2{}^b(\tau)]\langle\boldsymbol{\beta}; f[t] \,|\, \{\alpha_k{}^d + \kappa^{bd}_k(\tau)\}; t\rangle \tag{2.31}$$

In order to define the radiation power for transitions between the Fock states $|\mathbf{n}; \text{in}\rangle$ and $|\mathbf{m}; f\rangle$, we must take into consideration the fact that the amplitude $\langle\boldsymbol{\beta}; f|\boldsymbol{\alpha}; t\rangle$ is the generating function of the amplitudes $\langle\mathbf{m}; f|\mathbf{n}; t\rangle$ and the matrix elements of the Weyl operators $\langle\beta_j; f|D_j|\alpha_j; t\rangle$ are the generating functions of the matrix elements $\langle m_j; f|D_j|n_j; t\rangle$ (Malkin et al., 1971, 1973; Ivanova et al., 1975).

3. *Radiation of a Charged Particle in Homogeneous Varying Electric and Magnetic Fields*

As an example using the above–mentioned formulas, let us consider the radiation of the particle with charge $e(e > 0)$ and mass m moving in the time–dependent electromagnetic field with the potentials

$$\mathscr{A} = \tfrac{1}{2}[\mathscr{H}(t)\mathbf{r}], \qquad \varphi = -\mathscr{E}(t)\mathbf{r} \tag{3.1}$$

where

$$\mathscr{H}(t) = \{0, 0, \mathscr{H}(t)\}, \qquad \mathscr{E}(t) = \{\mathscr{E}_1(t), \mathscr{E}_2(t), 0\}, \qquad \mathscr{E}\mathscr{H} = 0$$

In calculating we use the results obtained in Malkin and Man'ko (1970)

Let us take it for granted that before the moment of time $t = 0$ the constant magnetic field $\mathscr{H}_{\text{in}}$ existed. At the zero moment the nonstationary electric field emerged, and the magnetic field began to vary. In the remote future the electric field will vanish, and the magnetic field will be equal to $\mathscr{H}_f = \text{const.}$

The Hamiltonian of the particle in the field (3.1) may be written in the form

$$H_0(t) = H_\perp(t) + H_z, \qquad [H_\perp(t), H_z] = 0$$

Here the Hamiltonian H_z describes the free motion of the charged particle along the Z axis. We designate the stationary orthonormalized states of H_z by $|n_3; t\rangle$. The Hamiltonian $H_\perp(t)$ describes the motion in the X-Y plane. It is a two-dimensional quadratic Hamiltonian.

Let us consider the motion in the X-Y plane in accord with the scheme obtained in Section 2, using the designations of this section. In Malkin and Man'ko (1970) the invariants $\underset{\sim}{A}_1(t)$ and $\underset{\sim}{A}_2(t)$ with the commutation relations of the boson operators have been constructed for the Hamiltonian $H_\perp(t)$:

$$\begin{aligned} \underset{\sim}{A}_1 &= 2^{-1/2}(\dot{\epsilon}\rho + i\epsilon\partial/\partial\rho^* + \dot{\epsilon}\rho_0 - \epsilon\dot{\rho}_0) \\ \underset{\sim}{A}_2 &= 2^{-1/2}(i\dot{\epsilon}\rho^* - \epsilon\partial/\partial\rho + i\dot{\epsilon}\rho_0^* - i\epsilon\dot{\rho}_0^*) \end{aligned} \tag{3.2}$$

In equation (3.2) the variable $\rho(t)$ is

$$\rho(t) = -\left(\frac{m}{2}\right)^{1/2} (x + iy) \exp\left(i\int_0^t \Omega \frac{d\tau}{2}\right), \qquad \Omega = \frac{e\mathscr{H}}{m} \tag{3.3}$$

The function $\epsilon(t)$ is determined as

$$\ddot{\epsilon} + \Omega^2 \frac{\epsilon}{4} = 0, \qquad \epsilon = |\epsilon| \exp\left(i\int_0^t |\epsilon|^{-2}\, d\tau\right) \tag{3.4}$$

The variables $\rho_0(t)$ and $\dot{\rho}_0(t)$ are

$$\begin{aligned} \rho_0 &= -2^{-1/2}(i\epsilon^*\gamma_1 + \epsilon\gamma_2^*) \\ \dot{\rho}_0 &= -2^{-1/2}(i\dot{\epsilon}^*\gamma_1 + \dot{\epsilon}\gamma_2^*) \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} &\gamma_1 = -\frac{1}{\sqrt{2}}\int_0^t \epsilon\Phi\, d\tau, \qquad \gamma_2 = -\frac{i}{\sqrt{2}}\int_0^t \epsilon\Phi^*\, d\tau \\ &\Phi = \frac{e}{(2m)^{1/2}}(\mathscr{E}_1 + i\mathscr{E}_2) \exp\left(i\int_0^t \Omega\frac{d\tau}{2}\right), \qquad \boldsymbol{\gamma} \equiv (\gamma_1, \gamma_2) \end{aligned} \tag{3.6}$$

In the paper by Malkin and Man'ko (1970) the coherent states $|\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2; t\rangle \equiv |\underset{\sim}{\boldsymbol{\alpha}}; t\rangle$ and the transition amplitudes between the coherent states $|\underset{\sim}{\boldsymbol{\alpha}}; t\rangle$ and $|\underset{\sim}{\boldsymbol{\beta}}; t\rangle$ were found:

$$\begin{aligned} \langle\underset{\sim}{\boldsymbol{\beta}}; f[t]\,|\underset{\sim}{\boldsymbol{\alpha}}; t\rangle = \xi^{-1} \exp\{&-2^{-1}(|\underset{\sim}{\boldsymbol{\alpha}}|^2 + |\underset{\sim}{\boldsymbol{\beta}}|^2 + |\boldsymbol{\gamma}|^2) + \xi^{-1}[(\underset{\sim}{\boldsymbol{\alpha}} - \boldsymbol{\gamma})\underset{\sim}{\boldsymbol{\beta}} + \eta^*(\underset{\sim}{\alpha}_1\underset{\sim}{\alpha}_2 \\ &+ \gamma_1\gamma_2 - \underset{\sim}{\alpha}_1\gamma_2 - \underset{\sim}{\alpha}_2\gamma_1) - \eta\underset{\sim}{\beta}_1^*\underset{\sim}{\beta}_2^*] + \underset{\sim}{\alpha}_1\gamma_1^* + \underset{\sim}{\alpha}_2\gamma_2^* \\ &+ i\int_0^t (\Phi\rho_0^* + \Phi^*\rho_0)\, d\tau\} \end{aligned} \tag{3.7}$$

$\underset{\sim}{\alpha}_j$ and $\underset{\sim}{\beta}_j$ being constant complex numbers, and the parameters $\xi(t)$ and $\eta(t)$ are expressed in terms of $\epsilon(t)$ and its derivative $\dot{\epsilon}(t)$ in the following way:

$$\begin{gathered} \epsilon = (2/\Omega_f)^{1/2}\,[\xi \exp(i\Omega_f t/2) - i\eta \exp(-i\Omega_f t/2)] \\ \dot{\epsilon} = i(\Omega_f/2)^{1/2}\,[\xi \exp(i\Omega_f t/2) + i\eta \exp(-i\Omega_f t/2)] \\ \epsilon^*\dot{\epsilon} - \epsilon\dot{\epsilon}^* = 2i, \qquad |\xi|^2 - |\eta|^2 = 1 \end{gathered} \tag{3.8}$$

We shall discuss the radiation for the transitions between the states $|\underset{\sim}{\alpha}, n_3; t\rangle \equiv |\underset{\sim}{\alpha}; t\rangle\, |n_3; t\rangle$ and $|\underset{\sim}{\beta}, n_3; t\rangle$. We stick to the case when $(\mathbf{e})_z = 0$ and $\mathbf{k}_z = 0$, i.e., we assume that a photon is emitted parallel to the X–Y plane.

According to the formulas of Section 2 the matrix element $J(\tau, t)$ corresponding to equation (2.30) is given by

$$J(\tau, t) = eV(\tau, t)\langle \underset{\sim}{\beta}; f[t]\, |\underset{\sim}{\alpha} + \boldsymbol{\varkappa}(\tau); t\rangle \exp\,[\psi_1(\tau) + \psi_2(\tau)], \qquad \boldsymbol{\varkappa} \equiv (\kappa_1, \kappa_2) \tag{3.9}$$

where

$$V(\tau, t) = \sum_{\mu=1}^{5} w_\mu(\tau, t) u_\mu(\tau) \tag{3.10}$$

$$\{u_\mu(\tau)\} = \{\underset{\sim}{\alpha} + \boldsymbol{\varkappa}(\tau), \underset{\sim}{\beta}^*, 1\} \tag{3.11}$$

$$\begin{aligned} \kappa_{1,2}(t) &= -\frac{\epsilon(t)}{2m^{1/2}}(\mathrm{k}_{x,y} + ik_{y,x}) \exp\left(\pm i \int_0^t \Omega \frac{d\tau}{2}\right) \\ \psi_1(t) &= -i\left(\frac{2}{m}\right)^{1/2} \mathrm{Im}\left[(k_y + ik_x)\rho_0 \exp\left(-i\int_0^t \Omega\frac{d\tau}{2}\right)\right] \\ \psi_2(t) &= i\mathrm{Im}\,[\boldsymbol{\varkappa}(t)\underset{\sim}{\alpha}^*] \end{aligned} \tag{3.12}$$

The values w_μ are

$$\begin{aligned} w_{1,2}(\tau, t) &= \frac{e_{y,x} + ie_{x,y}}{2m^{1/2}\xi(t)} \exp\left(\mp i \int_0^\tau \Omega \frac{d\tau_1}{2}\right)[\xi(t)\chi^*_{(\pm)}(\tau) - i\eta^*(t)\chi_{(\mp)}(\tau)] \\ w_{3,4}(\tau, t) &= \frac{e_{y,x} - ie_{x,y}}{2m^{1/2}\xi(t)} \exp\left(\pm i \int_0^\tau \Omega \frac{d\tau_1}{2}\right)\chi_{(\pm)}(\tau) \\ w_5(\tau, t) &= w_3(\tau, t)[\xi(t)\gamma_1^*(t) - \eta^*(t)\gamma_2(t)] + w_4(\tau, t)[\xi(t)\gamma_2^*(t) \\ &\quad - \eta^*(t)\gamma_1(t)] + 2\sqrt{2}\mathrm{Re}\left\{(e_x - ie_y)\exp\left(-i\int_0^\tau \Omega\frac{d\tau_1}{2}\right)\right. \\ &\quad \left. \times\, [\dot{\rho}_0(\tau) - i\Omega(\tau)\rho_0(\tau)/2]\right\} \qquad (e_{x,y} \equiv (\mathbf{e})_{x,y}) \end{aligned} \tag{3.13}$$

$\chi_{(\pm)}$ being defined as

$$\chi_{(\pm)}(\tau) = \dot{\epsilon}(\tau) \pm i\Omega(\tau)\epsilon(\tau)/2$$

So the radiation power of the particle in the fields (3.1) is given by the formula (2.24), where $J(\tau, t)$ is equation (3.9). In the dipole approximation it is necessary to consider $\varkappa = \psi_1 = \psi_2 = 0$ in the expressions (3.9) and (3.11).

Let us consider the case when at the moment of time $t = 0$ the vector potential (3.1) abruptly changes from the constant value $\mathscr{A}_{\text{in}} = (-\mathscr{H}_{\text{in}}y/2, \mathscr{H}_{\text{in}}x/2, 0)$ to the constant value $\mathscr{A}_f = (-\mathscr{H}_f y/2, \mathscr{H}_f x/2, 0)$, and the scalar potential (3.1) is equal to zero. Let the Glauber P functions be $P_f = 1/\pi^2, P_{\text{in}} = \delta^{(2)}(\underset{\sim}{\alpha} - \underset{\sim}{\alpha}_0)$, where $\underset{\sim}{\alpha}_0$ gives the initial $(t \to -0)$ coordinates x_0, y_0 and the velocities v_{x0}, v_{y0} of the particle which moves along the classical trajectory in the phase space. Summing over the polarization vectors and integrating with respect to the solid angle and the frequency, from the formula (2.24) we obtain the following expression for the dipole radiation power of the charge:

$$\mathscr{P} = (e^2\Omega_f{}^2/6\Omega_{\text{in}}^2)[(\Omega_f + \Omega_{\text{in}})^2(v_{x0}^2 + v_{y0}^2) + 2\Omega_{\text{in}}(\Omega_f{}^2 - \Omega_{\text{in}}^2)(v_{x0}y_0 - v_{y0}x_0)$$
$$+ \Omega_{\text{in}}^2(\Omega_f - \Omega_{\text{in}})^2(x_0{}^2 + y_0{}^2) + \Omega_{\text{in}}(\Omega_f - \Omega_{\text{in}})^2] \qquad (3.14)$$

The expression (3.14) in the approximation $|\underset{\sim}{\alpha}_{01}| \gg 1, |\underset{\sim}{\alpha}_{02}| \gg 1$ coincides with the expression for the radiation power, which can be obtained by calculating according to the formulas of classical electrodynamics. If the magnetic field is constant, we must take $\Omega_{\text{in}} = \Omega_f$.

4. *Conclusion*

In summing up we point out once more that using the coherent-state method has enabled us to calculate the radiation of quadratic systems to the first order in the magnitudes of the charges. In the formulas for the radiation power when the transitions between the coherent states are considered, it is easy to proceed to the classical limit.

As the Hamiltonian of the interaction of the radiation field with the charges in the dipole approximation is quadratic, we can calculate the dipole one-photon transition amplitudes exactly, i.e., without the series expansion in respect to the magnitudes of the charges.

Let us note also that from the one-photon transition amplitudes (2.30) calculated without being bound to the dipole approximation, it is easy to obtain multipole and dipole one-photon transition amplitudes by means of the series expansion in respect to the wave vectors $\mathbf{k}$.

The coherent-state method employed for evaluating the radiation of nonrelativistic quadratic systems can be easily generalized to relativistic quadratic systems by using the proper time representation. This possibility is connected with the fact that we can express the exponential factors $\exp(\pm i\mathbf{k}\mathbf{q}^a)$ in terms of the Weyl operators determining the coherent states.

It is noteworthy that the coherent-state method enables us to evaluate both the radiation for transitions between the energy eigenstates and the radiation of wave packets such as coherent states.

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